## MATHEMATICAL SCIENCES

# A Study on Transient and steady state behaviour of M/G/1 Retrial Queueing systems using Markov regenerative stochastic petri nets.

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## Abstract

In this paper we study the M/G/1 retrial queueing system which orbit capacity is limited. We also study the representation of this system using Markov regenerative stochastic petri nets. We also examine how this representation can be used to find the steady state & transient probabilities of the system.

**Keywords :** Retrial queueing system, transient & steady state probabilities, Markov regenerative stochastic petri nets.

**Introduction :** A retrial queueing system consists of a service station and an orbit as shown in the figure 1. Customers can arrive at the service station either from outside or from the orbit. If an arriving customer finds an idle server, he receives service immediately and leaves the system after service completion. However, if the arriving customer is not able to find an idle server, he is obliged to join the orbit. He can then repeat his request for service from the orbit after a random amount of time. From the orbit, he keeps trying for service until he finds an idle server. At this stage, he receives service and leaves the system after service completion. On the retrials, an orbiting customer is treated in the same way as a new arrival.

The consideration of repeated requests for service introduces great analytical difficulties in the study of queueing models. In fact, detailed analytical results exists only for some special retrial queueing systems. In these special systems, some assumptions are necessary on some characteristic of the queue such as the retrial time distribution, the population size, the number of servers etc. For other retrial queueing systems, the performance evaluation is limited to numerical algorithms and approximation methods

In this paper first we briefly review the basics of- stochastic petrinets. We also give a brief introduction to, the theory of Non Markovian SPN's and in particular of Markov regenerative SPN's. we give a representation for the *M/G/1retrial* queueing system as an MRSPN.

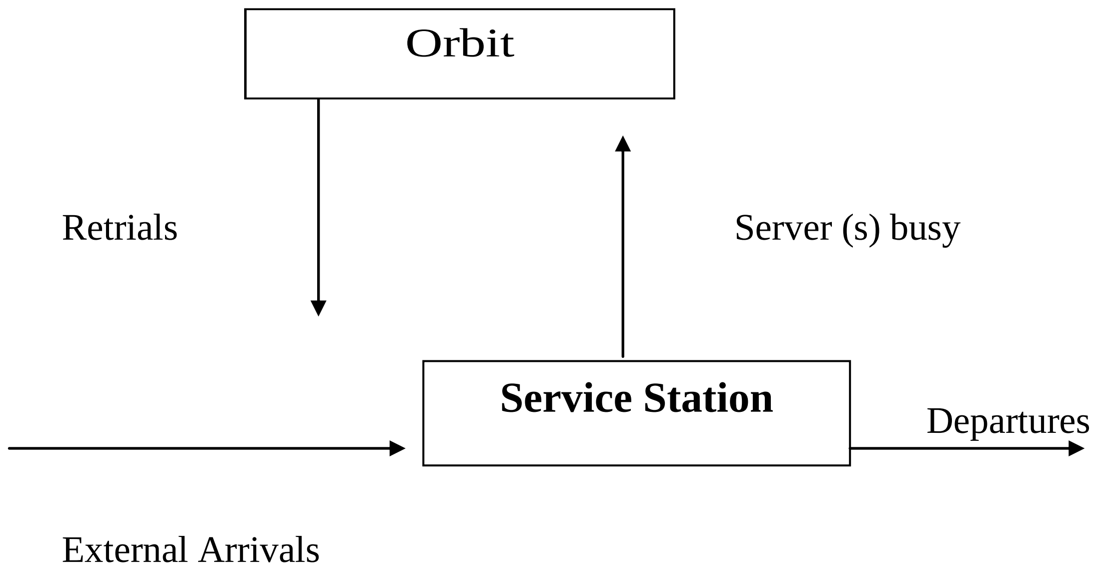


Fig. 1. Schematic diagram of a Retrial Queuing System

 

**Stochastic, Generalized & Markov Regenerative Petri Nets.:** About one decade ago, Molloy Natkin and Symons independently proposed associating exponentially distributed firing delays to transitions of a Petri net. Generalized Stochastic Petri Nets(GSPN), introduced by Ajmone Marsan, Balbo and Conte, relax this condition by, also following transition firings in constant zero time. Stochastic activity networks (SANs) and stochastic reward nets (SRNs) are two other classes of Petri nets in which transition firing is either exponentially distributed or constant zero. GSPNs, SANs, and SRNs are isomorphic to Continuous -Time Markov Chains( CTM Cs).

A SPN is called a Markov Regenerative Stochastic Petri Net I (MRSPN) if its marking process is a Markov Regenerative Process (MRGP) MRGPs are discrete-state, continuous-time stochastic processes with an embedded sequence of Regenerative Time Points, (RTP) at which the process enjoys the Markov property. Based on the concept of memory in a SPN, RTPs can be defined as follows:

A Regeneration Time Point (RTP) in the marking process { $M(t): t \geq O$} underlying an SPN is an instant of time where all the ,transitions do not have memory By the memory less property of the MRGP at the RTP, . the analysis of an MRSPN can be split into independent sub problems given by the subordinated processes between any two consecutive RTPs. The probability functions that must be evaluated for the transient analysis of a MRSPN are commonly referred to as the global and local kernels .

The global kernel $K(t) = [K_{ij}(t)]$ describes the occurrence of the next RTP:

$K_{ij}(t) = \text{prob}\{ M_1 = j, T_1 \leq t / M(O) = i\}$

M(O)=i indicates the initial condition for the marking process, $T_1$ is the next RTP and $M_I$ is the right continuous state hit by the marking process at the next RTP.

The local kernel $E(t) = [E_{ij}(t)]$ describes the state transition probabilities inside a regeneration period, before the next RTP occurs;

$Eij(t) = prob\{M(t)= j, T_1 > t/ M(O) = i\}$,

Using the theory of Markov Regenerative processes in Trivedi and Kulkarni__have studied the. transient and steady state behaviour of the MRSPN

**Transient Analysis of a MRSPN** *Let* $V_{ij}(t)=prob\{M(t)=j/M(O)=i\}$

$V(t) = [V_{ij}(t)]$

The transient analysis of the MRSPN is based on the following general renewal type equation.

$$V_{ij}(t) = E_{ij}(t) \int_o^t \sum_k dk_{ik}(y)V_{kj}(t-y)$$

o r

$$V_{ij}(t) = E_{ij}(t) + \sum K_{ik}(t)*V_{kj}(t)$$

In matrix notation, $V(t) = E\{t) + K(t)*V(t)$

**Steady State analysis of an MRSPN :** We consider the steady state analysis of an MRSPN whose underlying Marking process is finite and ergodic, so that the limiting probability distribution exist. These can be computed using the general theory of MRGP. We briefly review the theory below,

Define,. $\mu_m = E[T_1 / Y_o = m]$, *where*

$Y_o = M(0)$

The steady state probability vector by,

$$v = vP$$
$$\sum_{i\in\Omega} v_i = 1$$

v=(v $_i$) of the Embedded morkov chain is given

Define, $\alpha_{mn} = \int_{o}^{\infty} P\{M(t) = n, T_1 > t/Y_o = m\}dt = \int_{o}^{\infty} E_{m.n}(t)dt.$

Note that $\mu_m = \sum_{n\in\Omega(m)} \alpha_{mn}$, Where Ω (m) is the set of all states reachable from m before the next RTP occurs.

## Transient and steady state behaviour of M/G/1 Retrial Queueing systems using Markov regenerative stochastic petri nets.

We consider a single server retrial queueing system Customers are assumed to arrive in accordance with a Poisson process With parameter $\lambda$ An arriving customer, who finds an idle server, receives service immediately and leaves the system after completion of his service. The service times of customers at the server is assumed to be identically and independently distributed random variables with a distribution function *B(x)*.

If the arriving customer finds the server busy, he is obliged to leave the service station immediately and join the orbit, He then retries for service after a random delay. This random delay is assumed to be exponentially distributed with parameter $\mu$. If, after retrying for service, the customer is able to find an idle server he gets serviced immediately and leaves the system after service completion. We assume that the system has a population size N which can be as large as we please or may even be infinite.

We assume that the orbit has a finite capacity L, for the time being. Even this restriction can be removed by a suitable modification of the MRSPN model. We assume that inter arrival times, the intervals between repeated attempts (retrials) and the service times are mutually independent. Figure.2. shows the MRSPN model of this system.

The Place $P_a$ contains N tokens, which represents the condition that none of the N customers has arrived for service. The Place $P_s$ represents the condition that a customer is ready for service. The Place Po represents the orbit. .The initial marking of the net is given by Mo = {M ($P_a$ ), M ($P_s$), M ($P_o$)} = {N, 0, 0 } which represents the fact that none of the customers has arrived for service and the orbit is empty.

 

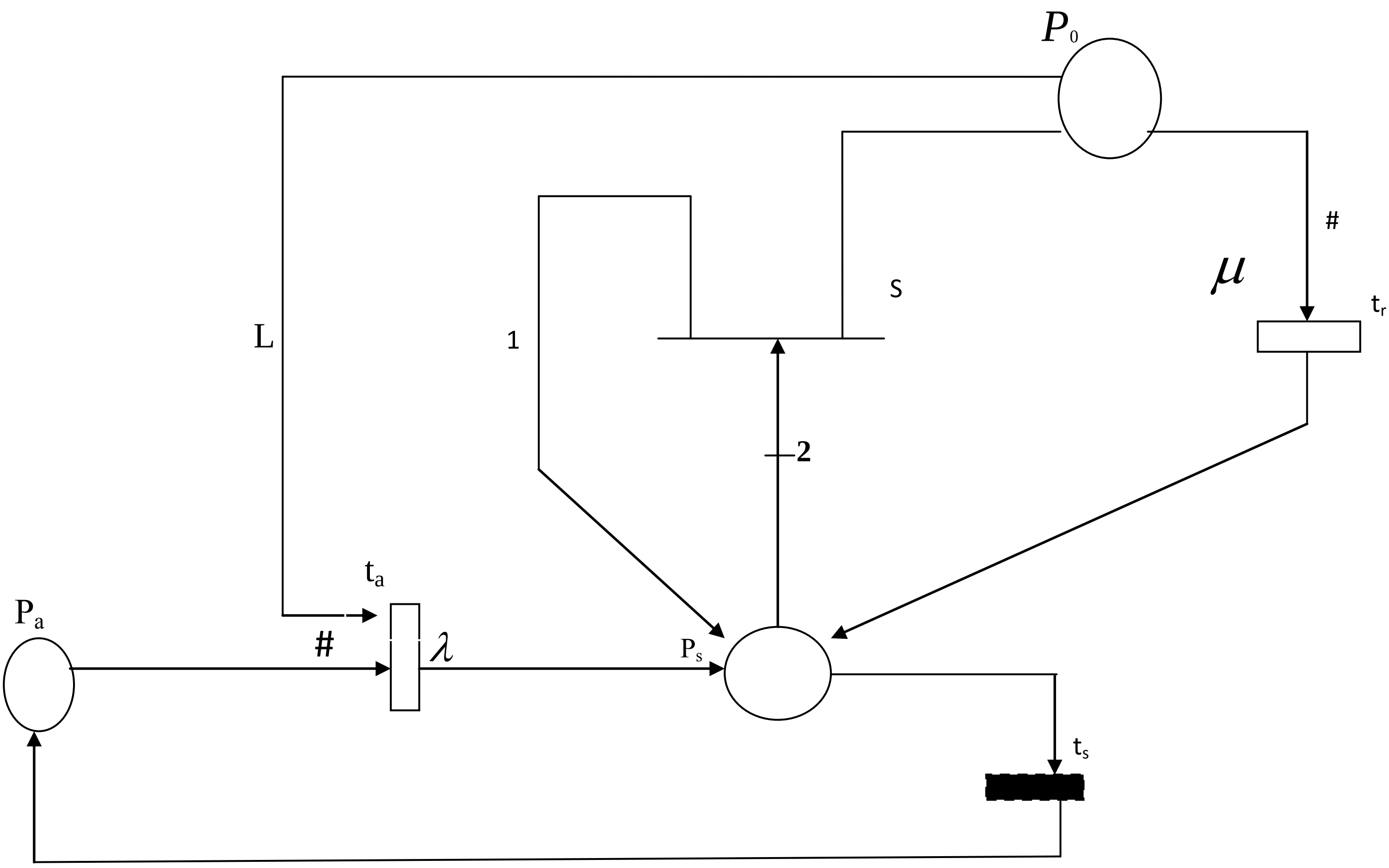


Fig.2. MRSPN for the *M/G/1* retrial queueing system

The transition $t_a$ when fired indicates the arrival of a new stomer. If the population is considered as a finite population, its firing rate is marking dependent. However. if we consider N to be sufficiently large or infinite, the transition firing rate is no longer marking dependant.

The immediate transition S is enabled when there are atleast 2 customers in the place $P_s$. thus indicating that the server busy. The firing of the immediate transition S thus enables the customer to leave thtservice area and enter into the orbit. This fact is represented by removing a token from Ps and placing a token in the orbit Po. Once in the orbit. the customers independently of each other try again for service, after a random delay that is exponentially distributed with a mean $1/\mu$.

When the timed transition $t_r$ fires. it represents the end of the waiting time of a customer in the orbit. One token is taken from $P_o$ and placed in $P_s$ . Thus. the firing rate of the transition $t_r$ wiIl depend upon the marking of the place $P_o$ . because a constant rate of retrial would imply that at a time. only one customer is allowed to retry. The general transition $t_s$ is enabled when the place $P_s$ contains at least one token. When It is fired. one token is deposited in $P_a$ which indicates that after service completion the customer has returned to the population.

The inhibitor arc from the place $P_o$ to $t_a$ has multiplicity L, thus indicating that the transition $t_a$ can fire and a new arrival can occur only when the number of tokens in the place Po is less than L. That is. the capacity of the orbit should not exceed L.

At any instant of time 't', the marking M(t) is given by the 3-tuple ($M_1$, $M_2$, $M_3$) where. $M_1$=Number of tokens in the place $P_a$.
$M_2$= Number of tokens in the place $P_s$ and $M_3$=Number of tokens in the place $P_o$.

We assume that the system starts empty at time t = 0 and that there are no customers in the orbit. Thus. $M_0 = (N.\ 0, 0)$ . The state space of the underlying marking process is thus given by

$$\Omega = \{(N-(r+s), r,s) / O \le r \le 1 \quad O \le s \le L\}.$$

In order to simplify the notations. we use the symbol (r, s) to denote the state $(N - (r + s), r, s)$. Since the only transition with a generally distributed firing time is the service transition $t_s$. the regeneration time points (RTP) are defined as follows:

1. Define $T_o$ as equal to zero.

2. If at $T_n$ the state of the system is (0, s), then $T_{n+1}$ will be the time at which either of the two EXP transitions $t_a$ or $t_r$ fires.

3. If at time $T_n$ the state of the system is (r, s) where $r \neq .$ 0, then $T_{n+1}$ is the time at which a service is completed, that *is*, transition $t_s$ fires.

Since the transition $t_s$ is never disabled by the firing of the two EXP transitions $t_r$ or $t_a$, ,preemption is not possible. Also, in each ma king at the most only one GEN transition fires therefore all the conditions of theory of MRSPN are satisfied. Before proceeding with the calculations of thelocal and global kernel, we give some definitions.

For a state (r, s) with $0 < r \leq 1$, ,we define $\Omega(r, s)$ =set of all states reachable from (r, s) by the firing of the exponential transition $t_a$ or $t_r$ before the before the next service completion.

Then,

For $r$ =1, $0 < s < L$,

$\Omega(1, s) = \{(1, s'): s < s' < L\}$

When a regeneration interval begins in the state *(r, s)* with $0 < r < 1$, the end of the interval occurs with a service completion (that is, a firing of the transition $t_s$). The marking process inside the regeneration interval is a CTMC with state space $\Omega(r, s)$ .This is called the process subordinated to the transition $t_s$.The infinitesimal generator matrix of this CTMC is denoted by *Q(r, s).*

For a state *(r', s')* not belonging to $\Omega((r, s))$ , the row corresponding to that state in Q(r, s) is a zero row. The other entries of Q(r, s) are as follows:

For $(r', s') \in \Omega(r, s)$ with $r' < 1$ $s' < L$, the corresponding row has three nonzero Entries in the columns corresponding to the states

$(r', s')$, $(r' + 1, s' - 1)$ *and* $(r' + 1, s')$. These nonzero entries are given by

$- [s'\mu + (N - (r' + s'))\lambda], s'\mu, [N - (r' + s')]\lambda$ respectively.

For $r' < 1$, $s' = L$ the row corresponding to the state $(r'\ s')$ has only two nonzero entries in the columns corresponding to $(r'\ L)$, $(r' + 1, L - 1)$. These nonzero entries are given by $- L\mu$ and $L\mu$ respectively.

For $r' = 1$, $s' \neq L$ the row corresponding to the state $(1.\ s')$ has only two nonzero entries in the columns corresponding to $(1, s')$ and $(1, s' + 1)$ . These nonzero entries are given by $- [N - (1 + s')]\lambda$, $[N - (1 + s')]\lambda$. The row corresponding to the state (1, L) is a zero row.

For $0 < r \leq 1$, define.

$\Omega_g(r, s)$ = set of all states reachable from (r, s) (not necessarily directly) after the firing of the GEN transition $t_s$ ( after the completion of a service).

$$\Omega_g(r, s) = \{(r' - 1, s') : (r', s') \in \Omega(r, s)\}$$

**The expressions for the local kernel are as follows:-**

When r=O.

$$E_{(o.\ s)(r',s')}(t) = 0, \text{ if } (r', s') \neq (0, s)$$

$$= e^{-[(N-s)\lambda + s\mu]t} \quad \text{if} \quad (r', s') = (0, s),\ s < L$$

$$= e^{-L\mu t} \quad \text{if} \quad (r', s') = (0, L)$$

When $r = 1$, $0 \leq s \leq L$

$$E_{(c.\ s)(r',s')}(t) = 0,\ r' \neq 1$$

$E_{(c.\ s)(r',s')}(t) = 0,\ if\ s' < s$

$=[e^{Q(1,s)t}]_{(1,s)(1,s')}(1\text{-}B(t))$ if $s \le s' \le L \quad s \ne L$

$= (1\text{-}B(t))$ if $s'=s=L$

**The expressions for the global kernel are as follows,**

For $0 < s < L,\ r = 0,$

$K(o.\ s)(r',s)(t) = 0$ for $(r', s') \ne (1, s)\ or\ (1, s\text{-}1)$

$$K_{(o,\ s)(t,\ s\text{-}1)}(t) = \frac{s\mu}{(N-s)\lambda + s\mu}\left(1 - e^{-[N\text{-}s)\lambda + s\mu]t}\right)$$

$$K_{(O,s)(I,s)}(t) = \frac{(N-s)\lambda}{(N-s)\lambda + s\mu}\left(1 - e^{-[N\text{-}s)\lambda + s\mu]t}\right)$$

For $s = 0,\ r = 0$

$K_{(0,0)(r',s')}(t) = 0$ for $(r', s') \ne (1, 0)$

$K_{(0,0)(1,0)}(t) = 1 - e^{-N\lambda t}$ *for* $(r', s') = (1, 0)$

For $r=0,\ s= L,$

$K_{(0,L)(r',s')}(t) = 0$ for $(r', s') \ne (1, L\text{-}1)$

$$K_{(0,L)(1,L-1)}(t) = 1-e^{-L\mu t} \qquad \text{for} \quad (r', s') = (1,L-1)$$

For $0 < r \leq 1$ $\quad o \leq s \leq L,$

$$K_{(r,s)(r',s')}(t) = 0 \qquad \text{for } (r', s') \notin \Omega_g(r, s)$$

$$K_{(r,s)(r',s')}(t) = \int_0^t [e^{Q(r,s)x}]_{(r,s)(r'+1,s')} dB(x) \qquad \textit{Otherwise.}$$

The expressions for the transient probabilities

$V_{(r,s)(r',s')}(t) = Prob\{M(t) = (r', s')/ M(O) = (r, s)\}$

can then be obtained by solving the integral equation

$$V_{(r,s)(r',s')}(t) = E_{(r,s)(r',s')}(t) + \sum_{(r',s')\notin\Omega} \int_0^i V_{(r,s)(r's')}(t-y) dK_{(r,s)(r',s')} - (y)$$

Another alternate way of obtaining the transient probabilities is to obtain the solution in the transform domain.

$V(t) = [V_{(r,s,)(r',s')}(t)]$

$E(t)=[E_{(r,s)(r',s')}(t)]$

$K(t) = [K_{(r.s)r',s')}(t)]$

The above integral equation can be written as

$V(t) = E(t) + K(t) *V(t)$

Passing to the transform domain, we obtain the solution

$$\hat{V}(\theta) = \hat{E}(\theta) + \bar{K}(\theta)\ (\theta)$$

$$\hat{V}(\theta) = [I - \bar{K}(\theta)]^{-1}\hat{E}(\theta)$$

Having obtained $\hat{V}_{(r,\ s)(r',\ s')}(\theta)$, by using well known inversion formulae for the Laplace transforms, we can obtain the values of the transient probabilities $V_{(r,\ s)(r',\ s')}$ (t), These transient probabilities can then be used to obtain the various performance measures of the system at any instant of time t.

To obtain the steady state solutions, we assume that the system is ergodic. We first calculate the one step transition probabilities of the embedded Markov chain at the regeneration time points. These are given by $K(\infty) = [K_{(r,\ s)(r'.\ s)}(\infty)]$ . From these, we obtain the steady state probabilities of the embedded Markov chain which are given by

the solution of the equation. v = $vK(\infty)$ where v is the row vector containing the generic element $v_{(r,\ s)}$. This is the steady state probability of the embedded Markov' chain to be in the state $(r, s) \in \Omega$ . We then calculate.

$$\alpha_{(r.\ s)(r',\ s')} = \int_{0}^{\infty} E_{(r.\ s)(r',\ s')}(t)dt$$

From these we can calculate

$$\mu(r, s) = \sum_{(r',s')\in\Omega} \alpha_{(r,\ s)(r',\ s')}$$

**The steady state probabilities of the retrial queueing system are then given by**

$$P_{(r,s)} = \frac{\sum_{(r',s')\in\Omega} v_{(r',s')}\alpha_{(r',s')(r,s)}}{\sum_{(r',s')\in\Omega} \mu_{(r',s')} v_{(r',s')}}$$

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